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CONTINUUM MODELS OF MATERIALS WITH BEAM-MICROSTRUCTURE

KIL SOO KIM
Mechanical Engineering Department, Naval Postgraduate School, Monterey, CA 93943, U.S.A.

and

ROBERT L. PIZIALI
Failure Analysis Associates, 2225 East Bayshore Road, Palo Alto, CA 94301, U.S.A.

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Abstract—In biomechanics, the adequate mechanical model of cancellous bones, which consist of beam-microstructures, becomes essential for understanding the cause and development of arthritis. Mechanical behavior of a body consisting of microstructures requires an extended continuum model. The present study shows that the micropolar continuum of the general form is an adequate analytic model of 3-D periodic beam structures belonging to the single-atom type. This continuum differs from the micropolar continua defined by Eringen due to the presence of high order terms in the inertia properties. It is also shown that the stress and couple stress defined in this micropolar continuum model have physical meanings which are related to the microstructure. The low frequency dynamic characteristics of continuum models are investigated by calculating the natural frequencies of free vibrations of bodies of beam-structured materials. The results show that the effect of the high order terms is significant.

INTRODUCTION

This paper is concerned with continuum approaches in the description of the dynamic behavior of materials consisting of beam-microstructures. This problem has considerable interest in biomechanics since the adequate mechanical model of cancellous bones in human vertebra or in hip and knee joints becomes essential for understanding the cause and development of lower-back pain and arthritis. Cancellous bones consist of a large number of microstructures which are beam structures with rigid joints and the size of a microstructure in these materials compared with the macrodimension of the bodies is not as small as those of ordinary materials. Therefore, their behavior sometimes deviates from that of a classical continuum. The same problem also occurs in the dynamic analysis of large space structures which consists of a large number of flexible beams and trusses arranged in regular patterns.

In the early 1960s, a brisk activity arose in the area of mechanics of generalized continua (Kroner (1968); Aero and Kuvshinskii (1961); Toupin (1962); Mindlin (1964); Palmov (1964); Eringen and Suhubi (1964); the original work was done by Cosserat and Cosserat (1909)]. They extended the classical theory of elasticity with the intention of accommodating the effect of microstructures in materials. However, for simple materials such as metals, no noticeable results due to the non-classical terms in theory of Cosserat-type continua were confirmed experimentally (Ellis and Smith, 1967; Perkins and Thomson, 1973; Gauthier and Jahaman, 1975). For fiber-reinforced or laminated composite materials, it was shown that the effective stiffness theory, which is a Cosserat-type continuum theory, describes the dispersive phenomena of propagating harmonic waves in an infinite medium, which do not appear in homogeneous classical materials (Herrmann and Achenbach, 1967; Sun *et al.*, 1968). It has been noted that the increase in the velocity of ultrasonic waves in human bones is much larger than that in simple viscoelastic solids. Yoon and Katz (1983) suggested that the additional dispersion can be explained by considering a bone to be a Cosserat continuum with microstructures.

It was considered early that mechanical behavior of 2-D planar rectangular grids can be described adequately by Cosserat-type continua, especially by micropolar continua

(Eringen, 1966). Banks and Sokolowski (1968) demonstrated a close analogy between the governing equations of the equivalent continuum of a planar rectangular grid obtained from force equilibrium conditions of a unit element, and that of the couple stress theory for orthotropic bodies. Askar and Cakmak (1968) considered a 2-D model composed of lumped particles joined by extensible and flexible massless rods as a model of micropolar continua, since the governing field equations of the model were similar to those of micropolar continua. Bazant and Christensen (1972) showed that a micropolar continuum is a continuum approximation of a large planar rectangular grid framework under initial axial forces. Sun and Yang (1973) obtained a continuum model using energy equivalence and variational principles in their study of the dynamics of planar rectangular gridworks. Kanatani (1979) obtained dispersion curves of shear waves in planar rectangular grid frameworks for a wide range of wave numbers from the equations of motion of a complex valued micropolar continuum model. Noor and Nemeth (1980a, b) developed micropolar beam models to analyze beamlike lattices with rigid joints. However, in these previous works, it was not clear whether the continuum models for these materials with beam-microstructure were always the same as that of Eringen's micropolar continuum theory. In addition, there is no consistent method for the determination of material properties of the continuum model and the effect of these non-classical parameters were not thoroughly investigated.

In the present study, the problem of defining the adequate continuum model of a 3-D periodic beam structure in the linear elastic range is investigated systematically using the crystal lattice concept (McKie and McKie, 1974). A given 3-D periodic beam structure belongs to either a monatomic or a polyatomic beam structure. It is shown that the adequate continuum model which describes the static behavior of any monatomic 3-D periodic beam structure is the same micropolar continuum defined by Eringen. However, the continuum model of a periodic beam structured material differs from Eringen's micropolar continua in the expression for the inertia property of the continuum: not only the micro-rotational inertia terms but also higher order terms are defined. The material constants can be calculated systematically from the material properties and geometry of the microstructure. Stresses and couple stresses defined in this continuum model have definite physical meaning which can be expressed by discrete forces and moments acting on the unit microstructure. In the final part of the paper, the influence of high order terms upon dynamic behavior of these materials is investigated.

THEORETICAL DEVELOPMENT

A 3-D periodic beam structure is characterized by its spatial periodicity. It can be constructed by repetitions of a unit structure. The distribution of joints has the unique spatial periodicity of the beam structure. The beam connections at one joint may be the same or different from the beam connection at another joint. Assuming that each set of joints of the same beam connections is each type of particle, the 3-D periodic beam structure can be considered as a crystalline solid consisting of these fictitious particles.

Interactions between these fictitious particles are defined from the structural properties of beams. A stiffness matrix and a mass matrix of a beam element based on beam theory approximately represent the structural properties of a beam. A simple beam element, which includes the effect of rotatory inertia and shear deformations, is used in the present study (Przemieniecki, 1968).

For a general lattice structure with N particles to each cell, each different particle in a cell can be distinguished by an index k, $k = 1, \ldots, N$. Choosing any one cell as a reference cell, we can label different cells by a triple lattice index $\mathbf{L}(L_1, L_2, L_3)$. A particle in a general lattice is specified by indices $\mathbf{L}$ and k. A fictitious particle in a crystal lattice model of a periodic beam structure has six degrees-of-freedom of motion in 3-D space. Therefore, a vector $\mathbf{u}(\mathbf{L}, k)$, which represents translational and rotational displacements of a joint, has six components

$$\mathbf{u}(\mathbf{L},k) = \begin{Bmatrix} u(\mathbf{L},k) \\ v(\mathbf{L},k) \\ w(\mathbf{L},k) \\ \varphi_1(\mathbf{L},k) \\ \varphi_2(\mathbf{L},k) \\ \varphi_3(\mathbf{L},k) \end{Bmatrix} \tag{1}$$

where, u, v, and w are the components of translation along the X-, Y-, and Z-coordinate axis, respectively, and φ_1, φ_2, and φ_3 are the rotation about the X-, Y-, and Z-coordinate, respectively.

Consider a system of n beams. The strain energy stored in the system of n beams can be expressed as follows assuming the deformations in the beams are those in the static case:

$$\text{S.E.} = 1/2\,\{u\}\,[K]\,\{u\} \tag{2}$$

where $[K]$ is the total stiffness matrix of the system and $\{u\}$ the displacement vector of joints.

The kinetic energy of the n beams can be expressed as follows:

$$\text{K.E.} = 1/2\,\{\dot{u}\}\,[M]\,\{\dot{u}\} \tag{3}$$

where $[M]$ is the total consistent mass matrix of the sysem and $\{\dot{u}\}$ the velocity vector of joints.

The distribution of translations and rotations of joints can be represented by continuous functions $\mathbf{u}_k(\mathbf{x}, t)$, such that

$$\mathbf{u}_k(\mathbf{x}, t) = \mathbf{u}(\mathbf{L}, k) \tag{4}$$

where $\mathbf{x}$ is the position of the joint which is the kth particle in the $\mathbf{L}(L_1, L_2, L_3)$ lattice.

For a periodic beam structure of the single-atom type, all joints are equivalent. Therefore, one continuous vector function $\mathbf{u}(\mathbf{x}, t)$ is necessary for the description of the displacement of the beam structure. Using the first-order Taylor series expansion of a continuous vector function $\mathbf{u}(\mathbf{x}, t)$ about $\mathbf{x} = \mathbf{x}_0$ as an approximate displacement vector of neighbor joints of the reference joint 0, the strain energy stored in n beams of the system, shown in Fig. 1 and expressed in eqn (2) by discrete variables, can be expressed by a continuous vector function and its first derivatives as follows:

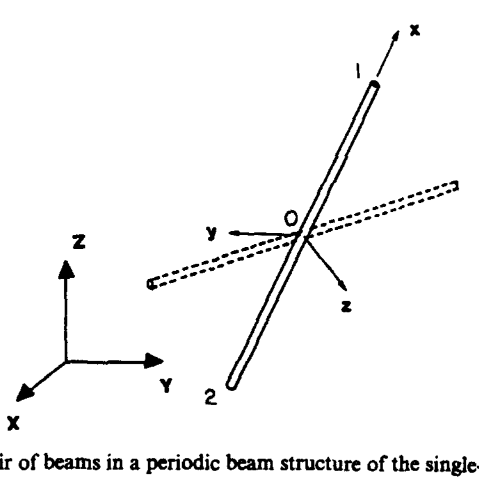

Fig. 1. A pair of beams in a periodic beam structure of the single-atom type.

$$\text{S.E.} = \frac{1}{2}\left\{\begin{matrix} \{u_0\} \\ \{u_0\} + \left(\Delta x_k \dfrac{\partial}{\partial x_k}\right)_1 \{u_0\} \\ \vdots \\ \{u_0\} + \left(\Delta x_k \dfrac{\partial}{\partial x_k}\right)_n \{u_0\} \end{matrix}\right\}^T [K] \left\{\begin{matrix} \{u_0\} \\ \{u_0\} + \left(\Delta x_k \dfrac{\partial}{\partial x_k}\right)_1 \{u_0\} \\ \vdots \\ \{u_0\} + \left(\Delta x_k \dfrac{\partial}{\partial x_k}\right)_n \{u_0\} \end{matrix}\right\}$$

$$= \frac{1}{2}\{u_0\}^T\left[\sum_{i=0}^{n}\sum_{j=0}^{n}[k]_{ij}\right]\{u_0\} + \{u_0\}^T\left\{\sum_{i=0}^{n}\left\{\sum_{j=1}^{n}[k]_{ij}\left[\left(\Delta x_k \frac{\partial}{\partial x_k}\right)_j\{u_0\}\right]\right\}\right\}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left(\Delta x_k \frac{\partial}{\partial x_k}\right)_i\{u_0\}\right\}^T [k]_{ij}\left\{\left(\Delta x_l \frac{\partial}{\partial x_l}\right)_j\{u_0\}\right\} \tag{5}$$

where

$$\{u_0\} = \{u(x_0, t)\}$$
$$[k] = [[k]_{ij}], \quad (i, j = 0, 1, \ldots, n)$$
$$\left(\Delta x_k \frac{\partial}{\partial x_k}\right)_j = \sum_{k=1}^{3} (\Delta x_k)_j \frac{\partial}{\partial x_k}$$

$(\Delta x_k)_j$ is the kth component vector, $(\mathbf{x}_j - \mathbf{x}_0)$, and $\mathbf{x}_j$ the position vector of the jth joint $(j = 0, 1, 2, \ldots, n)$.

The strain measures of micropolar continua are the asymmetric strain tensor ε_{kl} and the gradient of rotations γ_{kl}, which are defined as follows:

$$\varepsilon_{kl} = \frac{\partial u_l}{\partial x_k} + \varepsilon_{lkm}\phi_m \tag{6}$$

$$\gamma_{kl} = \frac{\partial \phi_k}{\partial x_l} \tag{7}$$

where ε_{lkm} is the alternating third-order tensor.

Then, the strain energy expression (5), which is a function of translational and rotational displacements and their first derivatives, reduces to the micropolar strain energy function $W(\varepsilon_{kl}, \gamma_{kl})$, as can be seen in the Appendix. W is a sum of a quadratic function of strain and a quadratic function of the gradient of rotations and no coupling terms appear for the beam structure of the single-atom type. Therefore, W is represented using only two fourth-order tensors as follows:

$$W = \tfrac{1}{2}(A_{klmn}\varepsilon_{kl}\varepsilon_{mn} + B_{klmn}\gamma_{kl}\gamma_{mn}). \tag{8}$$

The fourth-order tensors have the following major symmetry:

$$\begin{aligned} A_{klmn} &= A_{mnkl} \\ B_{klmn} &= B_{mnkl}. \end{aligned} \tag{9}$$

In a similar manner, the kinetic energy expression (3) reduces to the kinetic energy function $Y(\dot{u}_i, \dot{\phi}_i, \partial\dot{u}_i/\partial x_j, \partial\dot{\phi}_i/\partial x_j)$ where $\dot{u}_i$ $(i = 1, 2, 3)$ are components of the velocity vector and $\dot{\phi}_i$ are components of the angular velocity vector (see the Appendix). The kinetic energy function Y is a quadratic function not only of the velocity and angular velocity of a material point, but also of their first gradient. Y is represented in the following general form:

$$Y=\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\tfrac{1}{2}a_{ij}\dot{u}_i\dot{u}_j+\tfrac{1}{2}b_{ij}\dot{\phi}_i\dot{\phi}_j+c_{ij}\dot{u}_i\dot{\phi}_j\right)$$
$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(d_{ikj}\dot{u}_i\frac{\partial\dot{u}_j}{\partial x_k}+e_{ikj}\dot{u}_i\frac{\partial\dot{\phi}_j}{\partial x_k}+f_{ikj}\dot{\phi}_i\frac{\partial\dot{u}_j}{\partial x_k}+g_{ikj}\dot{\phi}_i\frac{\partial\dot{\phi}_j}{\partial x_k}\right)$$
$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{1}{2}h_{ijkl}\frac{\partial\dot{u}_i}{\partial x_j}\frac{\partial\dot{u}_k}{\partial x_l}+\frac{1}{2}j_{ijkl}\frac{\partial\dot{\phi}_i}{\partial x_j}\frac{\partial\dot{\phi}_k}{\partial x_l}+k_{ijkl}\frac{\partial\dot{u}_i}{\partial x_j}\frac{\partial\dot{\phi}_k}{\partial x_l}\right) \tag{10}$$

a_{ij} and b_{ij} are symmetric second-order tensors, and h_{ijkl} and j_{ijkl} are fourth-order tensors with major symmetry (see eqns (9)). Furthermore, from the characteristics of the mass matrix of beam pairs, the following conditions hold in the kinetic energy function Y:

$$\begin{aligned}a_{ij}&=\rho\delta_{ij}\\ c_{ij}&=0\\ d_{ijk}&=g_{ijk}=0\\ k_{ijkl}&=0\end{aligned} \tag{11}$$

where ρ is the apparent mass density of the continuum.

In the kinetic energy function of the micropolar continuum considered by Eringen (1966), only mass density ρ and the second-order term b_{ij} are defined and the higher order terms do not appear. In the classical continuum theory, mass density ρ is the only inertia property of the material. Therefore, the continuum model of the periodic beam-structured materials will be called the micropolar continuum of the general form.

Take T to be the total kinetic energy of the body and V to be the total potential energy of deformation of the body at time t. Then, T and V are the volume integrals of Y and W, respectively

$$T=\int_{\Omega}Y\left(\dot{u}_i,\dot{\phi}_i,\frac{\partial\dot{u}_i}{\partial x_j},\frac{\partial\dot{\phi}_i}{\partial x_j}\right)\mathrm{d}\Omega \tag{12}$$

$$V=\int_{\Omega}W(\varepsilon_{ij},\gamma_{ij})\,\mathrm{d}\Omega \tag{13}$$

where Ω is the volume of the body.

Let W_1 be the work done by the external forces and moments of the body. The variation of W_1 is given by

$$\delta W_1=\int_{\Omega}(b_i\delta u_i+B_i\delta\phi_i)\,\mathrm{d}\Omega+\int_s(T_i\delta u_i+m_i\delta\phi_i)\,\mathrm{d}s \tag{14}$$

where s is the boundary surface of the body, b_i the body force per unit mass, B_i the body couple per unit mass, T_i the surface traction acting on the boundary surface, and m_i the surface couple moment acting on the boundary surface. Hamilton's principle states that

$$\delta\int_t(T-V)\,\mathrm{d}t+\int_t\delta W_1\,\mathrm{d}t=0 \tag{15}$$

where the variation is taken between a fixed initial time t_0 and a final time t_1.

Define the stresses, σ_{ij}, and the couple stresses, m_{ij}, as follows:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{16}$$

$$m_{ij} = \frac{\partial W}{\partial \gamma_{ji}}. \tag{17}$$

Then, the equations of motion and the boundary conditions obtained from Hamilton's principle reduce exactly to those of linear micropolar elasticity considered by Eringen (1966),† except for the difference in the expression of inertia terms of the right-hand side, as follows:

equations of motion

$$\sigma_{ji,j} + b_i = \frac{\partial}{\partial t}\frac{\partial Y}{\partial \dot{u}_i} - \frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial t}\left(\partial Y \Big/ \partial\left(\frac{\partial \dot{u}_i}{\partial x_j}\right)\right)\right\} \tag{18}$$

$$m_{ji,j} + \varepsilon_{ijk}\sigma_{jk} + B_i = \frac{\partial}{\partial t}\frac{\partial Y}{\partial \dot{\phi}_i} - \frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial t}\left(\partial Y \Big/ \partial\left(\frac{\partial \dot{\phi}_i}{\partial x_j}\right)\right)\right\}; \tag{19}$$

boundary conditions

$$T_i^{(n)} = n_j\sigma_{ji} + n_j\frac{\partial}{\partial t}\left(\partial Y \Big/ \partial\left(\frac{\partial \dot{u}_i}{\partial x_j}\right)\right) \tag{20}$$

$$m_i^{(n)} = n_j m_{ji} + n_j\frac{\partial}{\partial t}\left(\partial Y \Big/ \partial\left(\frac{\partial \dot{\phi}_i}{\partial x_j}\right)\right) \tag{21}$$

where n_j is a component of the unit vector outward normal to the boundary surface.

The constitutive equations of the continuum model of a given periodic beam structure are obtained from eqns (16) and (17), using the strain energy function W. They are exactly the same as those of micropolar elasticity (Eringen, 1966) and can be expressed as follows:

$$\sigma_{kl} = A_{klmn}\varepsilon_{mn} \tag{22}$$

$$m_{kl} = B_{lkmn}\gamma_{mn}. \tag{23}$$

Stresses and the couple stresses in the micropolar continuum model of a periodic beam structure defined above have the following physical meanings (Kim, 1984)

$$\sigma_{kl} = \frac{1}{V^*}\sum_{i=1}^{n}(F_l^i \Delta x_i^k) \tag{24}$$

$$m_{kl} = \frac{1}{V^*}\sum_{i=1}^{n}(M_l^i \Delta x_i^k) \tag{25}$$

where V^* is the volume of a unit cell, F_1^i, F_2^i, F_3^i the components of a force acting on joint

† Eringen derived the equations of motion from balance laws. The balance laws included the conservation of micro-inertia since the micro-motion of a point particle in the micropolar continuum he assumed is completely independent to the macro-motion of the particle. However, the particles in the continuum model of periodic beam structure are connected such that the micro-motion is interrelated to the macro-motion by constitutive relations. Therefore, the momentum and the moment of momentum are balanced as total quantities and the micro-inertia defined in Eringen's micropolar continuum is not conserved independently in this continuum.

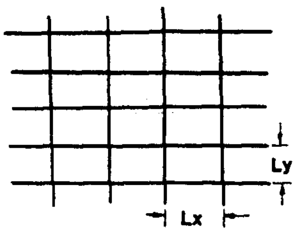

Fig. 2. A 2-D rectangular beam structure.

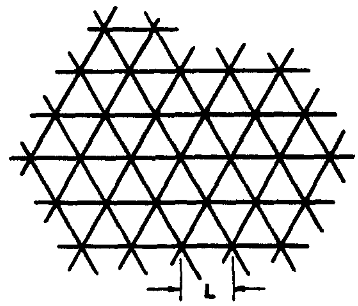

Fig. 3. A 2-D triangular beam structure.

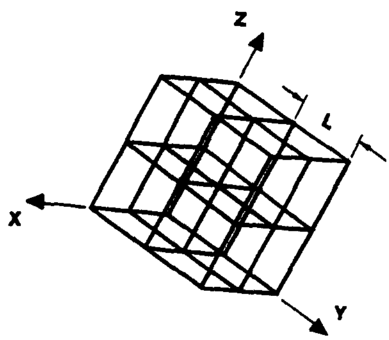

Fig. 4. A 3-D simple cubic structure.

i, M_1^i, M_2^i, M_3^i the component of a moment acting on joint i, and $(\Delta x_i^1, \Delta x_i^2, \Delta x_i^3)$ the components of the position vector of joint i relative to joint 0.

Relations (24) and (25) show that stresses and couple stresses are expressed by forces and moments acting on joints in the unit cell and are also dependent on the microstructure.

CALCULATION OF MATERIAL CONSTANTS

The computer program MAC (MAterial Constants) was developed to calculate material constants of micropolar continuum models of periodic beam structures. All of the following example beam structures are of the single-atom type and continuum models are orthotropic micropolar elastic solids.

(a) 2-D rectangular grid (Fig. 2, $L_x = 1.6$ mm, $L_y = 1.2$ mm);
(b) 2-D triangular grid (Fig. 3);
(c) 3-D simple cubic structure (Fig. 4).

The material properties of a unit beam member used are those of human compact bone and have the following values:

$$\begin{aligned} &\text{Young's modulus,} \quad E = 15.0 \text{ GPa} \\ &\text{Poisson's ratio,} \quad \nu = 0.3 \\ &\text{mass density,} \quad \rho = 1.8 \text{ g cm}^{-3}. \end{aligned} \tag{26}$$

A unit beam member has the geometric values which are typical values of human cancellous bones. Assuming the cross section of a beam is a circle, the shear coefficient k is taken as 0.886 in the calculation (Cowper, 1966). The thickness of the 2-D bodies is assumed to be the unit thickness

$$\begin{aligned} &\text{cross-sectional area,} && A = 0.0078\,\text{mm}^2 \\ &\text{second moment of inertia of the cross-section,} && I = 4.91\times10^{-6}\,\text{mm}^4 \\ &\text{length,} && L = 1.6\,\text{mm.} \end{aligned} \tag{27}$$

The micropolar continuum models of periodic beam structures obtained above have two kinds of material constants: one group is classical material constants (A_{ijkl}, ρ) and the other group is non-classical material constants (B_{ijkl}, etc.). The material constants obtained are represented in Tables 1–3. The effect of non-classical terms can be estimated using the following characteristic lengths:

characteristic length of geometry

$$L = \text{the length of a unit beam in a microstructure};$$

characteristic length of stiffness

$$L_s = \sqrt{(B_{3131}/A_{1111})};$$

characteristic lengths of inertia

$$L_b = \sqrt{(b_{33}/\rho)}$$
$$L_h = \sqrt{(h_{2121}/\rho)}.$$

These characteristic lengths are compared in Table 4. Since $L_s < L$ for the example

Table 1. Material constants of a micropolar elastic solid 2-D rectangular grid

Stiffness		Unit
A_{1111}	97.500	
A_{2222}	73.125	
A_{1122}, A_{1221}	0.0	MPa
A_{2121}	0.378	
A_{1212}	0.285	
B_{3131}	0.2439	MPa mm^2
B_{3232}	0.1820	
Inertia		
ρ	20.48	10^{-3} g cm^{-3}
b_{33}	0.2054	
$e_{312} = -e_{213}$	2.496	
$e_{123} = -e_{321}$	1.053	
$f_{213} = -f_{312}$	0.643	
$f_{321} = -f_{123}$	0.272	10^{-6} g mm^{-1}
h_{1111}	9.984	
h_{2222}	4.212	
h_{1122}	0.0	
h_{2121}	11.134	
h_{1212}	4.700	
j_{3131}	0.733	10^{-6} g mm
j_{3232}	0.174	

Table 2. Material constants of a micropolar elastic solid 2-D triangular grid

Stiffness		Unit
A_{1111}, A_{2222}	95.085	
A_{1122}, A_{1221}	31.572	MPa
A_{2121}, A_{1212}	31.942	
B_{3131}	0.3168	MPa mm^2
B_{3232}	0.3168	
Inertia		
ρ	30.40	10^{-3} g cm^{-3}
b_{33}	0.3744	
$e_{312} = -e_{213}$	3.242	
$e_{123} = -e_{321}$	3.242	
$f_{213} = -f_{312}$	0.836	10^{-6} g mm^{-1}
$f_{321} = -f_{123}$	0.836	
h_{1111}, h_{2222}	13.340	
h_{1122}	0.373	
h_{2121}, h_{1212}	14.090	
j_{3131}	0.952	10^{-6} g mm
j_{3232}	0.952	

Table 3. Material constants of a micropolar elastic solid 3-D simple cubic structure

Stiffness		Unit
$A_{1111}, A_{2222}, A_{3333}$	45.703	
$A_{1212}, A_{1313}, A_{2323}$ $A_{2121}, A_{3131}, A_{3232}$	0.134	MPa
$A_{1122}, A_{1133}, A_{2233}$ $A_{1221}, A_{1331}, A_{2332}$	0.0	
$B_{1111}, B_{2222}, B_{3333}$	0.0219	
$B_{1212}, B_{1313}, B_{2323}$ $B_{2121}, B_{3131}, B_{3232}$	0.1143	MPa mm^2
$B_{1122}, B_{1133}, B_{2233}$ $B_{1221}, B_{1331}, B_{2332}$	0.0	
Inertia		
ρ	16.45	10^{-3} g cm^{-3}
b_{11}, b_{22}, b_{33}	0.1420	
$e_{312} = -e_{213}$ $e_{123} = -e_{321}$ $e_{231} = -e_{132}$	1.70	
$f_{213} = -f_{312}$ $f_{321} = -f_{123}$ $f_{132} = -f_{231}$	0.302	10^{-6} g mm^{-1}
$h_{1111}, h_{2222}, h_{3333}$	4.680	
$h_{1212}, h_{1313}, h_{2323}$ $h_{2121}, h_{3131}, h_{3232}$	5.219	
$h_{1122}, h_{1133}, h_{2233}$ $h_{1221}, h_{1331}, h_{2332}$	0.0	
$j_{1111}, j_{2222}, j_{3333}$	0.0059	
$j_{1212}, j_{1313}, j_{2323}$ $j_{2121}, j_{3131}, j_{3232}$	0.3435	10^{-6} g mm
$j_{1122}, j_{1133}, j_{2233}$ $j_{1221}, j_{1331}, j_{2332}$	0.0	

Table 4. Comparison between characteristic lengths (unit = mm)

Microstructure	L	L_t	L_b	L_h	L_h/L_b
2-D Rectangle	1.6	0.05	0.10	0.737	7.37
2-D Triangle	1.6	0.058	0.11	0.681	6.14
3-D Simple cube	1.6	0.05	0.093	0.563	6.06

materials considered, the effect of non-classical terms B_{ijkl} in stiffness is very small. Therefore, in static problems, the effect of microstructures (i.e. deviation from classical continua) is restricted to very narrow boundary zones (Kim, 1984). The ratios L_b/L are also very small. However, L_h and L are of the same order of magnitude and L_h is about six times larger than L_b. Therefore, the effect of beam-microstructure will be the greatest in dynamic problems due to the fourth-order terms h_{ijkl} when the wavelength becomes the same order of magnitude as L_h. This is investigated in the following section.

FREE VIBRATION OF BODIES WITH PERIODIC BEAM STRUCTURES

To see the effect of the high order terms, h_{ijkl}, etc. on the dynamic behavior of periodic beam structured bodies, modal analyses of both micropolar continuum models and equivalent classical continuum models are performed.

This classical continuum was defined from the micropolar continuum using asymptotic expansions and it was shown that the solution of classical elasticity is the outer solution of the perturbation analysis of micropolar elasticity (Kim, 1984). The material constants, C_{ijkl}, of classical continuum can be derived directly from the material constants A_{ijkl} of the corresponding micropolar continuum.

Since it is very difficult to derive analytic solutions of vibration problems of anisotropic bodies due to the difficulty of satisfying boundary conditions, the finite element method is used in the modal analyses. A finite element formulation of the elastodynamic problem of micropolar continuum of the general form is presented in Kim (1984) and the difference between the two micropolar continua is clearly described. These finite element formulations are implemented by the computer program MICE (MICro Elasticity) described in Kim (1984).

Example problems

Natural frequencies and mode shapes of the lowest modes of undamped free vibration of the following bodies are calculated to compare the dynamic behavior of continuum models with those of exact periodic beam structures:

(a) a 2-D square with square grid structure (Fig. 5);
(b) a 2-D hexagon with triangular grid structure (Fig. 6);
(c) a 3-D cube with cubic lattice (Fig. 7).

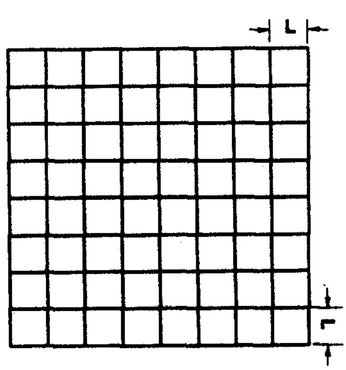

Fig. 5. A 2-D square with square grid microstructure.

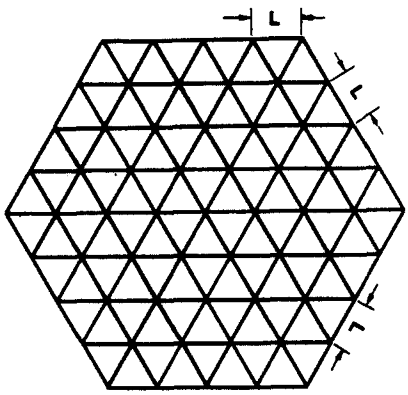

Fig. 6. A 2-D hexagon with triangular grid structure.

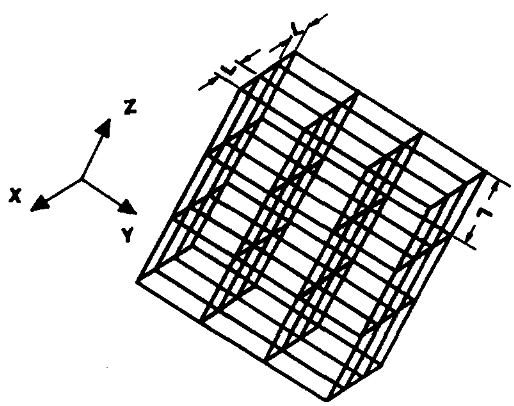

Fig. 7. A 3-D cube with cubic lattices.

The material properties and the geometric values of a unit beam member in the bodies are the same as the ones used in the calculation of material constants, except that the diameter of a unit beam member in the 2-D hexagon is doubled. Therefore, the material constants in Tables 1 and 3 are directly used in the analysis.

The beams in the 2-D bodies are confined to in-plane motion. Therefore, the flexural vibrations of the body are not allowed. Exact solutions are obtained using beam elements in the beam structural models and the maximum error is estimated to be less than 1.5% for the modes calculated.

Results and discussions

The 12, 9, and 12 modes of free vibration of the 2-D square, the 2-D hexagon, and the 3-D cube are calculated, respectively. They include the lowest to highest vibration modes. The exact solution and all solutions from continuum models give the same corresponding mode shapes for each mode. Therefore, the natural frequency for each mode can be compared. They are compared in Tables 5–7. They show that the micropolar continuum of the Eringen form has the highest natural frequency due to the lack of high order inertia terms. The micropolar continuum of the general form gives very accurate solutions. The classical continuum derived from the micropolar continuum also gives good results. This shows that the optimum classical continuum should be derived from the micropolar continuum.

Since the number of microstructures in the bodies analyzed are not large, the effect of boundary beams on natural frequencies is not small. To match the "full" beam boundary condition, the continuum model was corrected by attaching "half" beams on the boundary of continuum models.

Table 5. Natural frequencies of face shear modes of the 2-D square body (unit = 1 kHz)

Mode No.	Exact	Classical	Micropolar (general form)	Micropolar (Eringen form)
1	3.769	3.997	4.012	4.083
2, 3	5.872	6.265	6.165	6.535
4	8.612	9.213	8.806	10.018
5	9.545	10.188	9.804	11.146
6, 7	12.533	13.498	12.356	15.449
8	15.859	17.262	14.939	20.831
9	16.512	18.055	15.504	21.639
10, 11	19.844	21.943	17.670	27.582
12	23.016	25.151	19.517	33.174

Table 6. Natural frequencies of face shear modes of the 2-D hexagonal body (unit = 10 kHz)

Mode No.	Exact	Classical	Micropolar (general form)	Micropolar (Eringen form)
1, 2	6.214	6.782	6.511	6.788
3, 4	7.421	7.896	7.246	7.903
5	8.953	10.032	9.629	10.033
6	9.309	10.697	9.868	10.720
7	9.598	11.533	10.103	11.576
8, 9	10.596	12.184	10.459	12.203

Table 7. Natural frequencies of the lowest modes of the 3-D cubic body (unit = 1 kHz)

Mode No.	Exact	Classical	Micropolar (general form)	Micropolar (Eringen form)
1, 2	6.329	6.688	6.464	7.362
3, 4, 5	7.751	9.314	9.085	9.787
6	11.535	13.172	12.042	13.843
7, 8, 9	12.015	15.578	13.173	17.399
10, 11, 12	12.042	15.578	13.233	17.399

The errors in the natural frequencies of free vibration from modal analyses of continuum models and from modal analyses of corrected continuum models are plotted in Figs 8–10. The results show that the effect of the high order terms in the inertia property of the micropolar continuum model is significant for the problem considered. The improvement of the results with the correction of boundary conditions can also be seen in the figures. The boundary effects are larger for lower modes since the wavelengths of the low modes are comparable to the body lengths of the beam structured materials. For higher modes the wavelengths of vibration become smaller than the body length. Therefore, the boundary effect is reduced and the effect of the material properties, especially the high order inertia terms, become significant.

CONCLUSIONS

Periodic beam structures must be classified by their discrete models to define an adequate continuum model. For a periodic beam structure of the single-atom type, the micropolar continuum of the general form, defined in this study, is the adequate continuum model. Stress and couple stress defined in the micropolar continuum model of periodic beam structures of the single-atom type have definite relationships with forces and moments acting on neighbor joints of a unit structure. Therefore, the stresses and couple stresses at a point in a continuum model of a beam-structured material should be interpreted with the microstructure in mind.

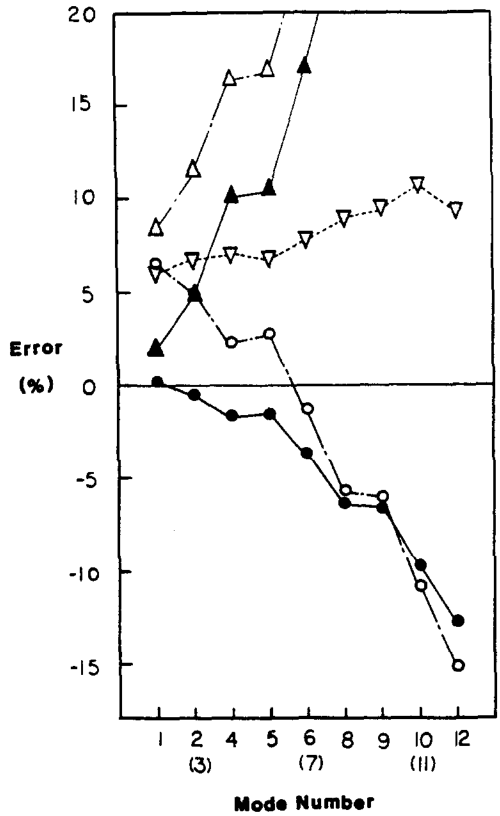

Fig. 8. Errors of natural frequencies of continuum models compared with exact solutions for the 2-D square: ▽--------, classical; △——·——, micropolar (Eringen); ○——·——, micropolar (general); ▲————, micropolar (Eringen)+boundary beams; ●————, micropolar (general)+boundary beams.

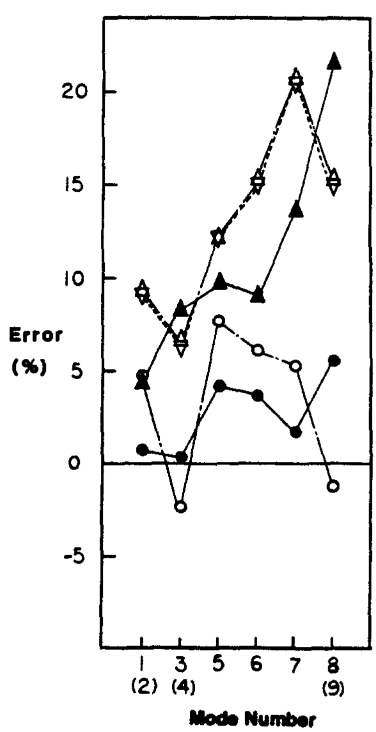

Fig. 9. Errors of natural frequencies of continuum models compared with exact solutions for the 2-D hexagon: ▽--------, classical; △——·——, micropolar (Eringen); ○——·——, micropolar (general); ▲————, micropolar (Eringen)+boundary beams; ●————, micropolar (general)+boundary beams.

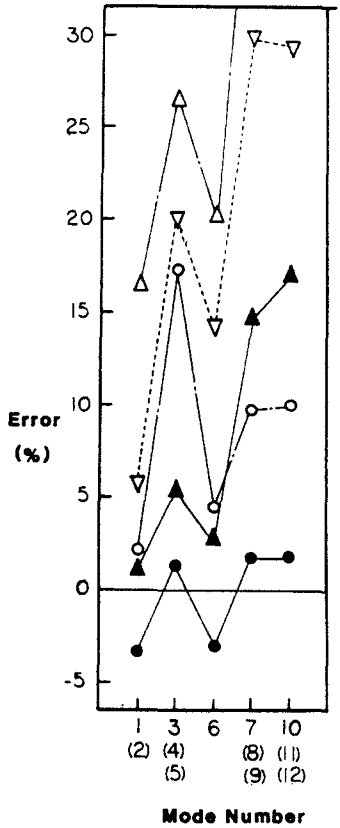

Fig. 10. Errors of natural frequencies of continuum models compared with exact solutions for the 3-D cube: ▽-------, classical; △——·——, micropolar (Eringen); ○——·——, micropolar (general); ▲————, micropolar (Eringen)+boundary beams, ●————, micropolar (general)+boundary beams.

Besides the mass density ρ and the micro-rotational inertia term b_{ij} higher order terms appear in the inertia property of the continuum model of a beam-structured material. The effect of these high order terms on the elastodynamic behavior of beam-structured materials is significant.

REFERENCES

Aero, E. L. and Kuvshinskii, E. V. (1961). Fundamental equations of the theory of elastic media with rotationally interacting particles. *Soviet Phys. Solid State* **2**, 1272–1281.

Askar, A. and Cakmak, A. S. (1968). A structural model of a micropolar continuum. *Int. J. Engng Sci.* **6**, 583–589.

Banks, C. B., Jr. and Sokolowski, M. (1968). On certain two-dimensional applications of the couple stress theory. *Int. J. Solids Structures* **4**, 15–29.

Bazant, Z. P. and Christensen, M. (1972). Analogy between micropolar continuum and grid frameworks under initial stress. *Int. J. Solids Structures* **8**, 327–346.

Cosserat, E. and Cosserat F. (1909). *Theorie des Corps Deformables*. Hermann, Paris, translated (1968) NASA-TT-F-11561, NASA, Washington, D.C.

Cowper, C. R. (1966). The shear coefficient in Timoshenko's beam theory. *ASME J. Appl. Mech.* **33**, 335–340.

Ellis, C. W. and Smith, C. W. (1967). A thin-plate analysis and experimental evaluation of couple-stress effects. *Expl Mech.* **7**, 371–380.

Eringen, A. C. and Suhubi, E. S. (1964). Nonlinear theory of simple micro-elastic solids. *Int. J. Engng Sci.* **2**, 189–203 and 389–404.

Eringen, A. C. (1966). Linear theory of micropolar elasticity. *J. Math. Mech.* **15**, 909–923.

Gauthier, R. D. and Jahaman, W. E. (1975). A quest for micropolar elastic constants. *ASME J. Appl. Mech.* **42**, 369–374.

Herrmann, G. and Achenbach, J. D. (1967). On dynamic theories of fiber-reinforced composites. *Proc. AIAA/ASME Eighth Structure Structural Dynamics Materials Conf.*, Palm Springs, California, pp. 112–118.

Kanatani, K. I. (1979). A micropolar continuum model for vibrating grid frameworks. *Int. J. Engng Sci.* **17**, 409–418.

Kim, K. S. (1984). Static and dynamic characteristics of materials with beam-microstructure. Ph.D. Dissertation, Stanford University.

Kroner, E. (Editor) (1968). *Proceedings of the IUTAM Symposium on Mechanics of Generalized Continua.* Springer, New York.
McKie, D. and McKie, C. (1974). *Crystalline Solids.* Wiley, New York.
Mindlin, R. D. (1964). Micro-structure in linear elasticity. *Archs Ration. Mech. Analysis* **16**, 51–78.
Noor, A. K. and Nemeth, M. P. (1980a). Micropolar beam models for lattice grids with rigid joints. *Comp. Meth. Appl. Mech. Engng* **21**, 249–263.
Noor, A. K. and Nemeth, M. P. (1980b). Analysis of spatial beamlike lattices with rigid joints. *Comp. Meth. Appl. Mech. Engng* **24**, 35–59.
Palmov, V. A. (1964). Fundamental equation of the theory of asymmetric elasticity. *PMM* **28**, 401–408.
Perkins, R. W. and Thomson, D. (1973). Experimental evidence of a couple-stress effect. *AIAA J.* **11**, 1053–1055.
Przemieniecki, J. S. (1968). *Theory of Matrix Structural Analysis.* McGraw-Hill, New York.
Sun, C. T., Achenbach, J. D. and Herrmann, G. (1968). Continuum theory for a laminated body. *ASME J. Appl. Mech.* **35**, 467–475.
Sun, C. T. and Yang, T. Y. (1973). A continuum approach toward dynamics of gridworks. *ASME J. Appl. Mech.* **40**, 186–192.
Toupin, R. A. (1962). Elastic materials with couple-stresses. *Archs Ration. Mech. Analysis* **11**, 385–414.
Yoon, H. S. and Katz, J. L. (1983). Is bone a Cosserat solid? *J. Mater. Sci.* **18**, 1297–1305.

APPENDIX: ENERGY FUNCTIONS

Strain energy function

Consider a beam pair which is a system of two equal beams connected at joint 1 as shown in Fig. 1. The strain energy stored in this beam pair can be expressed as follows using local coordinate variables:

$$\text{S.E.} = \frac{1}{2}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix}_l^T \begin{bmatrix} [\mathbf{k}_{00}] & [\mathbf{k}_{01}] & [\mathbf{k}_{02}] \\ [\mathbf{k}_{10}] & [\mathbf{k}_{11}] & [0] \\ [\mathbf{k}_{20}] & [0] & [\mathbf{k}_{22}] \end{bmatrix}_l \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix}_l \tag{A1}$$

where

$$[\mathbf{k}_{00}]_l = [k_{11}]_l + [k_{22}]_l$$
$$[\mathbf{k}_{01}]_l = [\mathbf{k}_{20}]_l = [k_{12}]_l$$
$$[\mathbf{k}_{02}]_l = [\mathbf{k}_{10}]_l = [k_{21}]_l = [k_{12}]_l^T$$
$$[\mathbf{k}_{11}]_l = [k_{22}]_l$$
$$[\mathbf{k}_{22}]_l = [k_{11}]_l$$

$[k_{11}]_l$, $[k_{12}]_l$, $[k_{22}]_l$, and $[k_{21}]_l$ form the stiffness matrix of a beam element and $\{u_i\}$ is the displacement vector of joint i.

The first-order Taylor series expansions for $\{u_1\}_l$ and $\{u_2\}_l$ about joint 0 are

$$\begin{aligned} \{u_1\}_l &= \{u_0\}_l + L\frac{\partial}{\partial x}\{u_0\}_l \\ \{u_1\}_2 &= \{u_0\}_l - L\frac{\partial}{\partial x}\{u_0\}_l \end{aligned} \tag{A2}$$

where L is the length of a beam.

Inserting eqns (A2) into eqn (A1) and after rearrangement, the strain energy becomes

$$\begin{aligned} \text{S.E.} = {} & \{u_0\}_l^T[[k_{11}]_l + [k_{22}]_l + [k_{12}]_l + [k_{21}]_l]\{u_0\}_l + L\{u_0\}_l^T[[k_{12}]_l - [k_{21}]_l + [k_{22}]_l - [k_{11}]_l]\frac{\partial}{\partial x}\{u_0\}_l \\ & + \tfrac{1}{2}L^2\left(\frac{\partial}{\partial x}\{u_0\}_l\right)^T[[k_{22}]_l + [k_{11}]_l]\frac{\partial}{\partial x}\{u_0\}_l. \end{aligned} \tag{A3}$$

Inserting the detailed expression of submatrices into eqn (A3) simplifies the strain energy expression

$$\begin{aligned} \text{S.E.} = {} & \frac{12EI_z}{L^3(1+\Phi_y)}\left(\frac{\partial v}{\partial x} - R\right)_l^2 + \frac{12EI_y}{L^3(1+\Phi_z)}\left(\frac{\partial w}{\partial x} + Q\right)_l^2 + \frac{EA}{L}\left(\frac{\partial u}{\partial x}\right)_l^2 + \frac{GJ}{L}\left(\frac{\partial P}{\partial x}\right)_l^2 \\ & + \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)}\left(\frac{\partial Q}{\partial x}\right)_l^2 + \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}\left(\frac{\partial R}{\partial x}\right)_l^2 \end{aligned} \tag{A4}$$

where (u, v, w) are the components of the translational displacement vector in local coordinates, (P, Q, R) the components of rotation about the x-, y-, and z-axes, E is Young's modulus of the beam, G the shear modulus of the beam, A the area of the beam cross section, J the polar moment of inertia of the beam cross section, I_y, I_z the second moments of inertia of the beam cross section about the y- and z-axis

$$\Phi_y = (12EI_z)/(k_z GAL_2)$$
$$\Phi_z = (12EI_y)/(k_y GAL_2)$$

and k_y, k_z the shear coefficients of the beam cross section

Using the strain measures defined in eqns (6) and (7), the strain energy stored in a beam pair can be expressed as follows:

$$\text{S.E.} = \frac{12EI_z}{L^3(1+\Phi_y)}(\varepsilon_{12})_l^2 + \frac{12EI_y}{L^3(1+\Phi_z)}(\varepsilon_{13})_l^2 + \frac{EA}{L}(\varepsilon_{11})_l^2 + \frac{GJ}{L}(\gamma_{11})_l^2 + \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)}(\gamma_{21})_l^2 + \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}(\gamma_{31})_l^2. \tag{A5}$$

The special characteristic of this expression is that there is no coupling term between the strain and the gradient of rotation.

If n pairs of beams are connected to joint 0, the strain energy stored in each pair of beams is expressed by eqn (A5) in each local coordinate. These strain energies can be summed when the strain measures in eqn (A5) are expressed in global coordinates by a coordinate transformation matrix, $[\lambda_{ij}]$, as follows:

$$\begin{aligned}(\varepsilon_{ij})_l &= \lambda_{ik}\lambda_{jl}(\varepsilon_{kl})_G \\ (\gamma_{ij})_l &= \lambda_{ik}\lambda_{jl}(\gamma_{kl})_G.\end{aligned} \tag{A6}$$

The strain energy function of the periodic structure is obtained when the sum of strain energies is divided by the volume of a unit cell. The strain energy function is the sum of a quadratic function of ε_{ij} and a quadratic function of γ_{ij}, and can be expressed using two fourth-order tensors as in eqn (8).

Kinetic energy function

After a similar procedure, the kinetic energy of a pair of beams, shown in Fig. 1, is expressed as follows:

$$\begin{aligned}\text{K.E.} = \rho AL\Bigg\{&(\dot{u}^2+\dot{v}^2+\dot{w}^2)+\frac{J}{A}\dot{P}^2+L^2\left(\frac{1}{210}+\frac{I_y}{5AL^2}\right)\dot{Q}^2 \\ &+L^2\left(\frac{1}{210}+\frac{I_z}{5AL^2}\right)\dot{R}^2+\frac{L^2}{6}\left(\dot{w}\frac{\partial\dot{Q}}{\partial x}-\dot{v}\frac{\partial\dot{R}}{\partial x}\right) \\ &+L^2\left(\frac{9}{210}+\frac{2I_y}{5AL^2}\right)\dot{Q}\frac{\partial\dot{w}}{\partial x}-L^2\left(\frac{9}{210}+\frac{2I_z}{5AL^2}\right)\dot{R}\frac{\partial\dot{v}}{\partial x} \\ &+\frac{L^2}{3}\left(\frac{\partial\dot{u}}{\partial x}\right)^2+L^2\left(\frac{13}{35}+\frac{6I_z}{5AL^2}\right)\left(\frac{\partial\dot{v}}{\partial x}\right)^2+L^2\left(\frac{13}{35}+\frac{6I_y}{5AL^2}\right)\left(\frac{\partial\dot{w}}{\partial x}\right)^2+L^2\frac{J}{3A}\left(\frac{\partial\dot{P}}{\partial x}\right)^2 \\ &+L^4\left(\frac{1}{105}+\frac{2I_y}{15AL^2}\right)\left(\frac{\partial\dot{Q}}{\partial x}\right)^2+L^4\left(\frac{1}{105}+\frac{2I_z}{15AL^2}\right)\left(\frac{\partial\dot{R}}{\partial x}\right)^2\Bigg\}\end{aligned} \tag{A7}$$

in local coordinate variables.

The kinetic energy expression, eqn (A7), can be expressed in terms of variables in global coordinates after a coordinate transformation of the velocity components and their first derivatives. If n pairs of beams are connected to joint 0, all the kinetic energies can be summed in global coordinates. Then, the kinetic energy function, which is a function of velocity, angular velocity, and their gradients, is obtained after dividing by the volume of the unit cell

$$Y = Y\left(\dot{u}_i, \dot{\varphi}_i, \frac{\partial\dot{u}_i}{\partial x_j}, \frac{\partial\dot{\varphi}_i}{\partial x_j}\right) \tag{A8}$$

where $\dot{u}_i$ are the velocity components in global coordinates and $\dot{\varphi}_i$ the angular velocity components in global coordinates.

One special characteristic of eqn (A7) is that there is no coupling term between velocity and angular velocity, between gradient of velocity and gradient of angular velocity, between velocity and gradient of velocity, and between angular velocity and gradient of angular velocity. Therefore, the kinetic energy function is simplified by these conditions (11).